Journal of Mathematical Sciences: Advances and Applications
Volume 3, Number 1, 2009, Pages 101-107

A PRACTICAL APPROACH TO SOLVE COUPLED SYSTEMS OF NONLINEAR PDE's

ALVARO SALAS and CESAR GÓMEZ

Department of Mathematics
Universidad de Caldas
Universidad Nacional de Colombia
Manizales, Colombia
e-mail: ahsalass@unal.edu.co

Department of Mathematics
Universidad Nacional de Colombia
Sede Bogotá
Colombia
e-mail: cagomezsi@unal.edu.co

Abstract

In this paper, we present the tanh method to obtain exact solutions to coupled MkDV system. This method may be applied to a variety of coupled systems of nonlinear ordinary and partial differential equations.

1. Introduction

The search of exact solutions to coupled systems of nonlinear partial differential equations is of great importance, because these systems appear in complex physics phenomena, mechanics, chemistry, biology and engineering. A variety of powerful and direct methods have been developed in this direction. The principal objective of this paper, is to

2000 Mathematics Subject Classification: 65Lxx.

Keywords and phrases: coupled MkdV equations, tanh method, Mathematica.

Received January 28, 2009

present the application of tanh method in solving coupled systems of two equations. The tanh method, developed by Malfliet in [2], proved to be effective and reliable for several nonlinear problems.

This method works effectively if the equation involves sine, cosine, hyperbolic sine, hyperbolic cosine functions, and exponential functions.

2. The Tanh Method

Consider a system of two coupled PDE's in the variables t, x

$$\begin{cases} P(u, v, u_x, v_x, u_t, v_t, u_{xt}, v_{xt}, u_{xx}, v_{xx}, \ldots) = 0, \\ Q(u, v, u_x, v_x, u_t, v_t, u_{xt}, v_{xt}, u_{xx}, v_{xx}, \ldots) = 0. \end{cases} \tag{2.1}$$

Using the wave transformation

$$u(x, t) = u(\xi), v(x, t) = v(\xi),$$

$$\xi = x + \lambda t, \tag{2.2}$$

where λ ia a constant, system (2.1) reduces to a system of two ordinary nonlinear differential equations

$$\begin{cases} p(u(\xi), u'(\xi), v''(\xi), \ldots) = 0, \\ q(v(\xi), v'(\xi), v''(\xi), \ldots) = 0. \end{cases} \tag{2.3}$$

The tanh method [2] is based on the idea of looking for solutions to system (2.3) in the form

$$u(\xi) = \sum_{i=0}^{m} a_i \varphi^i(\xi), \; v(\xi) = \sum_{j=0}^{n} b_j \varphi^j(\xi), \tag{2.4}$$

where the new variable $\varphi = \varphi(\xi)$ satisfies the Riccati equation

$$\varphi' = \varphi^2 + k, \tag{2.5}$$

whose solutions are given by

$$\varphi(\xi) = \begin{cases} -1/\xi & k = 0 \\ \sqrt{k}\tan(\sqrt{k}\xi) & k > 0 \\ -\sqrt{k}\cot(\sqrt{k}\xi) & k > 0 \\ -\sqrt{-k}\tanh(\sqrt{-k}\xi) & k < 0 \\ -\sqrt{-k}\coth(\sqrt{-k}\xi) & k < 0. \end{cases} \tag{2.6}$$

The integers m and n can be determined balancing the highest derivative terms with nonlinear terms in (2.3). Substituting (2.4) alongwith (2.5) into (2.3) and collecting all terms with the same power φ^i, we get two polynomials in the variable φ. Equating the coefficients of these polynomials to zero, we obtain a system of algebraic equations, from which the constants $a_i, b_j, \mu, \lambda, k$ are obtained explicitly. This allows us to obtain solutions to System (2.1).

3. Exact Solutions for Coupled MkdV Equations

The system of coupled MkdV equations [3] reads

$$\begin{cases} u_t = \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\eta u_x, \\ v_t = -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\eta v_x, \end{cases} \tag{3.7}$$

where η is a constant. We apply the transformation

$$u(x, t) = u(\xi),\ v(x, t) = v(\xi),\ \xi = x + \lambda t. \tag{3.8}$$

After substitution of (3.8) into (3.7) we get the following system of nonlinear ode's :

$$\begin{cases} 2(3\eta + \lambda)u' + 6u^2u' - 6u'v - 6uv' - 3v'' - u''' = 0, \\ (\lambda - 3\eta)v' - 3u^2v' + 3vv' + 3u'v' + v''' = 0. \end{cases} \tag{3.9}$$

We now substitute (2.4) into (3.9) and using (2.5) we obtain a system in the variable φ :

$$\begin{cases} c_1\varphi^{m+n+1} + c_2\varphi^{3m+1} + c_3\varphi^{m+3} \\ + c_4\varphi^{m+2} + c_5\varphi^{m+1} + c_6 = 0. \\ d_1\varphi^{2m+n+1} + d_2\varphi^{m+n+2} + d_3\varphi^{2n+1} \\ + d_4\varphi^{n+3} + d_5\varphi^{n+1} + d_6 = 0. \end{cases} \tag{3.10}$$

Balancing the highest order terms in (3.10) gives

$$\begin{cases} 3m + 1 = m + n + 1, \\ 2m + n + 1 = m + n + 2. \end{cases}$$

And, then $m = 1$ and $n = 2$. Therefore we seek solutions to (3.9) in the form

$$u(\xi) = a_0 + a_1\varphi(\xi),$$

$$v(\xi) = b_0 + b_1\varphi(\xi) + b_2\varphi^2(\xi). \tag{3.11}$$

Substituting (3.11) into (3.9), and using (2.5) we obtain an algebraic system in the variables $a_0, a_1, b_0, b_1, b_2, k$ and λ. Solving it with the aid of *Mathematica* we get the following solutions :

• First Family : For $\lambda = k + 3a_0^2$, $a_{-1} = -1$, $a_0 = a_0$, $b_0 = \eta$, $b_1 = -2a_0$ and $b_2 = 0$:

$$\begin{cases} u_1 = a_0 + \sqrt{-k}\tanh(\sqrt{-k}(x + (k + 3a_0^2)t), \\ v_1 = \eta + 2a_0\sqrt{-k}\tanh(\sqrt{-k}(x + (k + 3a_0^2)t)), \end{cases}$$

$$\begin{cases} u_2 = a_0 + \sqrt{-k}\coth(\sqrt{-k}(x + (k + 3a_0^2)t), \\ v_2 = \eta + 2a_0\sqrt{-k}\coth(\sqrt{-k}(x + (k + 3a_0^2)t)), \end{cases}$$

$$\begin{cases} u_3 = a_0 - \sqrt{k}\tan(\sqrt{k}(x + (k + 3a_0^2)t), \\ v_3 = \eta - 2a_0\sqrt{k}\tan(\sqrt{k}(x + (k + 3a_0^2)t)), \end{cases}$$

$$\begin{cases} u_4 = a_0 - \sqrt{k}\cot(\sqrt{k}(x + (k + 3a_0^2)t), \\ v_4 = \eta - 2a_0\sqrt{k}\cot(\sqrt{k}(x + (k + 3a_0^2)t)). \end{cases}$$

• Second Family : For $\lambda = -7k$, $a_0 = 0$, $a_1 = 3$, $b_0 = -\frac{10k}{3}$, $b_1 = 0$ and $b_2 = 2$:

$$\begin{cases} u_5 = 3\sqrt{-k}\tanh(\sqrt{-k}(x - 7kt)), \\ v_5 = \eta - \frac{10k}{3} - 2k\tanh^2(\sqrt{-k}(x - 7kt)), \end{cases}$$

$$\begin{cases} u_6 = 3\sqrt{-k}\coth(\sqrt{-k}(x - 7kt)), \\ v_6 = \eta - \frac{10k}{3} - 2k\coth^2(\sqrt{-k}(x - 7kt)), \end{cases}$$

$$\begin{cases} u_7 = 3\sqrt{k}\tan(\sqrt{k}(x - 7kt)), \\ v_7 = \eta - \frac{10k}{3} + 2k\tan^2(\sqrt{k}(x - 7kt)), \end{cases}$$

$$\begin{cases} u_8 = 3\sqrt{k}\cot(\sqrt{k}(x - 7kt)), \\ v_8 = \eta - \frac{10k}{3} + 2k\cot^2(\sqrt{k}(x - 7kt)). \end{cases}$$

• Third Family : $\lambda = -2k$, $a_0 = 0$, $a_1 = -2$, $b_0 = \eta$, $b_1 = 0$, $b_2 = 2$:

$$\begin{cases} u_9 = 2\sqrt{-k}\tanh(\sqrt{-k}(x - 2kt), \\ v_9 = \eta - 2k\tanh^2(\sqrt{-k}(x - 2kt)), \end{cases}$$

$$\begin{cases} u_{10} = 2\sqrt{-k}\coth(\sqrt{-k}(x - 2kt), \\ v_{10} = \eta - 2k\coth^2(\sqrt{-k}(x - 2kt)), \end{cases}$$

$$\begin{cases} u_{11} = -2\sqrt{k}\tan(\sqrt{k}(x-2kt), \\ v_{11} = \eta + 2k\tan^2(\sqrt{k}(x-2kt)), \end{cases}$$

$$\begin{cases} u_{12} = -2\sqrt{k}\cot(\sqrt{k}(x-2kt), \\ v_{12} = \eta + 2k\cot^2(\sqrt{k}(x-2kt)). \end{cases}$$

Figures 1 and 2 show graphics of u_{11} and v_{11}.

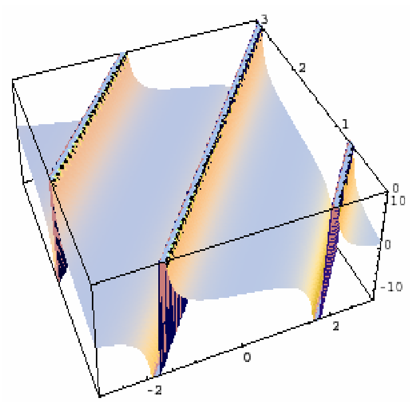

Figure 1. Graphic of function u_{11} for $k = 0.8$, $0 \le t \le 3$ and $-3 \le x \le 3$.

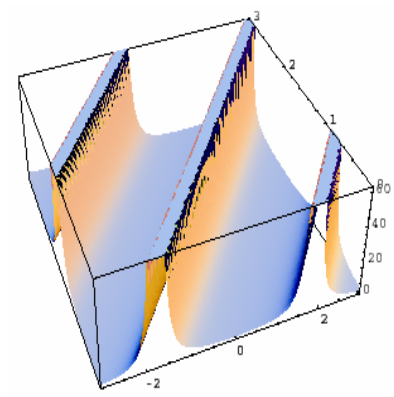

Figure 2. Graphic of function v_{11} for $k = 0.8$, $\eta = 1$, $0 \le t \le 3$ and $-3 \le x \le 3$.

• Fourth Family : $\lambda = k,\ a_0 = 0,\ a_1 = -1,\ b_0 = \eta - 2k,\ b_1 = 0,\ b_2 = -2$:

$$\begin{cases} u_{13} = \sqrt{-k}\tanh(\sqrt{-k}(x+kt), \\ v_{13} = \eta - 2k + 2k\tanh^2(\sqrt{-k}(x+kt)), \end{cases}$$

$$\begin{cases} u_{14} = \sqrt{-k}\coth(\sqrt{-k}(x+t), \\ v_{14} = \eta - 2k + 2k\coth^2(\sqrt{-k}(x+kt)), \end{cases}$$

$$\begin{cases} u_{15} = -\sqrt{k}\tan(\sqrt{k}(x+kt), \\ v_{15} = \eta - 2k - 2k\tan^2(\sqrt{k}(x+kt)), \end{cases}$$

$$\begin{cases} u_{16} = -\sqrt{k}\cot(\sqrt{k}(x+kt)), \\ v_{16} = \eta - 2k - 2k\cot^2(\sqrt{k}(x+kt)). \end{cases}$$

4. Conclusions

By using the tanh method [2], we obtained sixteen solutions to the coupled MkdV equations. The method is applicable to other coupled systems. There are other methods to solve nonlinear differential equations, for example, the tanh-coth method [1] and the projective Riccati equation method [4], [5].

References

[1] C. Gómez, Special forms of the fifth-order KdV equation with new periodic and soliton solutions, Appl. Math. Comput. 189(2) (2007), 1066-1077.

[2] W. Malfliet, Solitary wave solutions of nonlinear wave equations. Amer. J. Phys. 60(7) (1992), 650-654.

[3] K. R. Raslan, Int. J. Comput. Math. 81 (2004), 1497.

[4] A. Salas, Some solutions for a type of generalized Sawada-Kotera equation, Appl. Math. Comput. 196(2) (2008), 812-817.

[5] A. Salas and C. Gómez, El software Mathematica en la búsueda de soluciones exactas de ecuaciones diferenciales no lineales en derivadas parciales, mediante el uso de la ecuación de Riccati. Memorias Primer Seminario Internacional de tecnolog ías en educación matemática, Universidad Pedagógica Nacional, Bogotá, Colombia 1 (2005), 379-387.

■